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Functional relations in Stokes multipliers and solvable models related to $U_q(A_n^{(1)})$

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Abstract. Recently, Dorey and Tateo have investigated functional relations among Stokes multipliers for a Schrödinger equation (second-order differential equation) with a polynomial potential term in view of solvable models. Here we extend their studies to a restricted case of $(n+1)$th-order linear differential equations.

1. Introduction

In some remarkable papers [1, 2], Dorey and Tateo find marvellous connections between a Schrödinger equation with a polynomial potential term and solvable models. This success integrates two ingredients: the exact WKB analysis [3–10], on one hand, and the development in studies of solvable models and of the solvability structure in (perturbed) conformal field theory (CFT) [11–14] via the thermodynamic Bethe ansatz [16, 17] and nonlinear integral equations, on the other hand [18–22].

In view of the eigenvalue problem in quantum mechanics, $\mathcal{H}\Psi = E\Psi$, the quantity of interest is the spectral determinant, $\det(\mathcal{H} - E)$. For the potential term x^{2M} with $M =$ integer, it coincides with the 'vacuum expectation value' of a particular member of a fusion transfer matrix in a certain field-theory possessing $U_q(\widehat{\mathfrak{sl}_2})$ [1, 23]. The rest of the members in the fusion hierarchy are identified with the Stokes multipliers and their generalization [2] in the vacuum sector. Thus the result provides a unified view of Stokes multipliers and the spectral determinant.

The spectral determinant for the wider class of potentials $x^{2M} + \ell(\ell+1)/x^2$ with M general can be treated within a framework using Baxter's Q-operator and the quantum Wronskian relation [24].

We extend a part of studies in [2] to the higher-order differential equation case,

$$\partial^{n+1} y + (-1)^n (x^m + \lambda^{n+1}) y = 0 \tag{1}$$

with m an integer.

The $\lambda = 0$ case is essentially equivalent to the Turrittin equation of which solutions are calculated in terms of Meijier G-functions. There have been several results on this case [25–27].

We consider the Stokes multipliers associated with equation (1). They satisfy a set of functional relations in the complex λ-plane. We will show that non-trivial solutions to relations are expressible by the quantum Jacobi–Trudi formula which appears in fusion transfer matrices related to $U_q(A_n^{(1)})$ [28, 29].

We note the very recent result in [30] where the eigenvalue problem of equation (1) is argued for the $n = 2$ case with $x \in [0, +\infty]$ though a related but different method.

2. Asymptotic expansion and Stokes coefficients

We refer readers to [31] for the background on the subject.

Let us first discuss the asymptotic behaviour of a slightly generalized differential equation,

$$\begin{aligned} &\partial^{n+1} y + (-1)^n P(x) y = 0 \\ &P(x) = \sum_{j=0}^{m} a_j x^{m-j} \end{aligned} \tag{2}$$

where a_j are complex numbers and $a_0 = 1$. $P(x)$ will be referred to as the potential term in analogy with Schrödinger equations†. The factor $(-1)^n$ is not essential. It can be adsorbed into a redefinition of the angle of x and $a_{j\geqslant 1}$. For later convenience, we include this factor throughout this paper.

Now that $x = \infty$ is an irregular singular point of the equation, analytic properties of solutions depend on sectors in the complex x-plane. Let $\mathcal{S}_k$ be a sector in the plane satisfying

$$|\arg x - k\theta| \leqslant \tfrac{1}{2}\theta$$

for $x \in \mathcal{S}_k$, where $\theta = \frac{2\pi}{m+n+1}$. We then study the asymptotic behaviour of a subdominant solution in $\mathcal{S}_0$. Following [33, 34], we define b_h $(h = 1, 2, \ldots)$ by the relation,

$$\left(1 + \sum_{k=1}^{m} a_k x^{-k}\right)^{1/(n+1)} = 1 + \sum_{h=1}^{\infty} b_h x^{-h}.$$

A key function $E(x, \boldsymbol{a})$ is determined by b_h,

$$\begin{aligned} E(x, \boldsymbol{a}) &:= \int \left(1 + \sum_{h=1}^{h_m} b_h x^{-h}\right) x^{m/(n+1)} \mathrm{d}x \\ &= \frac{n+1}{m+n+1} x^{(m+n+1)/(n+1)} + \sum_{h=1}^{h_m} \frac{b_h}{m/(n+1) - h + 1} x^{m/(n+1)+1-h} \end{aligned}$$

where $h_m = N$ for $m = N(n+1) - j$, $(j = 1, \ldots, n)$. Here $\boldsymbol{a}$ denotes $(a_1, a_2, \ldots, a_m)$

In addition, we introduce an exponent ν_m by

$$\nu_m = \begin{cases} \frac{1}{2} nm & \text{for} \quad m \neq 0 \bmod n+1 \\ \frac{1}{2} nm + (n+1) b_{h_m+1} & \text{for} \quad m = 0 \bmod n+1. \end{cases} \tag{3}$$

Then we have a theorem,

Theorem 1. *In $\mathcal{S}_0$, there exists a subdominant solution $y(x, \boldsymbol{a})$ to equation (2) which admits the following asymptotics:*

$$\begin{aligned} &y(x, \boldsymbol{a}) \sim x^{-\nu_m/(n+1)} \mathrm{e}^{-E(x,\boldsymbol{a})} \\ &\partial^j y(x, \boldsymbol{a}) \sim x^{(jm-\nu_m)/(n+1)} \mathrm{e}^{-E(x,\boldsymbol{a})} \qquad j \geqslant 1. \end{aligned} \tag{4}$$

† In the case of a singularity at infinity of rank 1, i.e. $P(x) = p_0 + p_1/x + \cdots$ in the above, asymptotic solutions and numerical algorithms are discussed in [32] for a wider class of higher-order differential equations.

The range of the existence of the asymptotic expansion (4) (forgetting the subdominance) actually extends over $\mathcal{S}_0$. The precise determination of the range, however, poses a non-trivial problem even for $n = 1$ case. See the elaborate discussions in sections 9 and 10 of [34]. The straightforward extension of the argument may lead to the conclusion that $y(x, \boldsymbol{a})$ admits the asymptotic expansion (4) in an open sector,

$$\arg x < \frac{n+2}{m+n+1}\pi. \tag{5}$$

See the appendix.

The intriguing feature in the differential equation (2) is the following symmetry in rotating the x-plane.

Theorem 2. *Denote the above solution by $y(x, \boldsymbol{a})$. Then*

$$y_k(x, \boldsymbol{a}) := y(xq^{-k}, G^{(k)}(\boldsymbol{a}))q^{nk/2} \tag{6}$$

is also a solution to equation (2).

The parameter q signifies $\exp(\mathrm{i}\theta) = \exp\left(\mathrm{i}\frac{2\pi}{m+n+1}\right)$. The operation $G^{(k)}(\boldsymbol{a})$ is defined by $G^{(k)}(\boldsymbol{a}) = G^{(1)}(G^{(k-1)}(\boldsymbol{a}))$, $k \geqslant 2$ and $G^{(1)}(\boldsymbol{a}) = (a_1/q, a_2/q^2, \ldots, a_m/q^m)$.

A fundamental system of solutions (FSS) in $\mathcal{S}_k$ is formed by $(y_k, y_{k+1}, \ldots, y_{k+n})$. This is shown for the Turrittin equation in [27]. In general, it is not simple to show this directly. We use the range of the validity of the asymptotic expansion in equation (5) and justify the linear independence as follows.

We introduce a $(n+1) \times (n+1)$ matrix $\Phi_k(x)$ and the Wronskian $W_k := \det \Phi_k(x)$

$$\Phi_k(x) := \begin{pmatrix} y_k, & y_{k+1}, & \ldots, & y_{k+n} \\ \partial y_k, & \partial y_{k+1}, & \ldots, & \partial y_{k+n} \\ \vdots & & & \vdots \\ \partial^n y_k, & \partial^n y_{k+1}, & \ldots, & \partial^n y_{k+n} \end{pmatrix}. \tag{7}$$

Then equation (5) implies that y_{k+j}, $j = 0, \ldots, n$ has the asymptotic expansion (6) in the closed sector $\mathcal{S}_{k+(n+1)/2} \cup \mathcal{S}_{k+(n-1)/2}$. As W_k is constant in x, one then verifies the linear independence of these solutions by using the asymptotic expansions (4) and (6) in $\mathcal{S}_{k+(n+1)/2} \cup \mathcal{S}_{k+(n-1)/2}$. See discussions in [34] for $n = 1$ and [30] for $n = 2$

We stress that the argument below depends on the existence of $\Phi_k(x)$ as an FSS, rather than (5).

A matrix of Stokes multipliers $S_k^{(1)}$ connects FFS of $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$

$$\Phi_k(x) = \Phi_{k+1}(x)S_k^{(1)}. \tag{8}$$

The linear independence of solutions fixes S_k in the following form:

$$S_k^{(1)} = \begin{pmatrix} \tau_k^{(1)}, & 1, & 0, & 0, & \ldots, & 0 \\ \tau_k^{(2)}, & 0, & 1, & 0, & \ldots, & 0 \\ \vdots & & & & & \vdots \\ \tau_k^{(n)}, & 0, & 0, & 0, & \ldots, & 1 \\ \tau_k^{(n+1)}, & 0, & 0, & 0, & \ldots, & 0 \end{pmatrix}. \tag{9}$$

By Cramer's formula, one represents $\tau_k^{(j)}$ as

$$\tau_k^{(j)} = \frac{1}{W_k}\det\begin{pmatrix} y_{k+1}, & y_{k+2}, & \ldots, & y_k, & \ldots, & y_{k+n+1} \\ \vdots & & & & & \vdots \\ \partial^n y_{k+1}, & \partial^n y_{k+2}, & \ldots, & \partial^n y_k, & \ldots, & \partial^n y_{k+n+1} \end{pmatrix}. \quad (10)$$

The column vector, $(y_k, \partial y_k, \ldots, \partial^n y_k)$ is inserted in the jth column in the matrix of the denominator. Especially,

$$\tau_k^{(n+1)} = (-1)^n W_{k+1}/W_k. \quad (11)$$

Then we restrict ourselves to the case of interest (1). We find that $b_{h_m+1} = 0$ and thus $\nu_m = nm/2$ for any m. Under the operation of $G^{(1)}$, $G^{(1)}(a_m) = a_m q^{-m} = a_m q^{n+1}$. As $a_m = \lambda^{n+1}$, this means $G^{(1)}(\lambda) = \lambda q$. Thus a function in the kth sector has an argument λq^k.

3. Recursion relations

By the commensurability of the cone angle of $\mathcal{S}_k$ to 2π, the product of successive $m + n + 1$ Stokes matrices must be a unit matrix [33] modulo a sign factor due to the normalization in (6),

$$S^{(1)}_{m+n} S^{(1)}_{m+n-1} \cdots S^{(1)}_0 = (-1)^n I. \quad (12)$$

Regarding as a function of λ, $S^{(1)}_k = S^{(1)}(\lambda q^k)$. Thus equation (12) reads,

$$S^{(1)}(\lambda q^{m+n}) S^{(1)}(\lambda q^{m+n-1}) \cdots S^{(1)}(\lambda) = (-1)^n I. \quad (13)$$

The same relation can be recapitulated in terms of a generalized Stokes matrix $S^{(\ell)}_k$ connecting Φ_k and $\Phi_{k+\ell}$ [2, 31],

$$\Phi_k = \Phi_{k+\ell} S^{(\ell)}_k. \quad (14)$$

Similar to the above, $S^{(\ell)}_k = S^{(\ell)}(\lambda q^k)$ as a function of λ.

Equation (13) is obviously rewritten as

$$S^{(m+n+1)}(\lambda) = (-1)^n I. \quad (15)$$

By the above definition, we have two equivalent recursion relations

$$S^{(\ell+1)}(\lambda) = S^{(1)}(\lambda q^\ell) S^{(\ell)}(\lambda) \quad (16)$$

$$= S^{(\ell)}(\lambda q) S^{(1)}(\lambda). \quad (17)$$

We denote the (i, j) component of $S^{(\ell)}(\lambda)$ by $S^{(\ell)}_{i,j}(\lambda)$.

The main problems in this paper are:

(a) the expression of $S^{(\ell)}_{i,j}(\lambda)$ in terms of $\tau^{(a)}(\lambda)$s;
(b) restrictions imposed on $\tau^{(a)}(\lambda)$s by functional relations among them.

The first recursion relation (16) imposes the relations

$$S^{(\ell+1)}_{k,j}(\lambda) = \tau^{(k)}(q^\ell \lambda) S^{(\ell)}_{1,j}(\lambda) + S^{(\ell)}_{k+1,j}(\lambda) \qquad (1 \leqslant k \leqslant n + 1) \quad (18)$$

where formally we put $S^{(\ell)}_{n+2,j}(\lambda) = 0$.

The second recursion relation (17) yields

$$S_{k,1}^{(\ell+1)}(\lambda) = \tau^{(1)}(\lambda)S_{k,1}^{(\ell)}(\lambda q) + \tau^{(2)}(\lambda)S_{k,2}^{(\ell)}(\lambda q) + \cdots + \tau^{(n+1)}(\lambda)S_{k,n+1}^{(\ell)}(\lambda q) \quad (19)$$

and

$$S_{i,j+1}^{(\ell+1)}(\lambda) = S_{i,j}^{(\ell)}(q\lambda) \qquad (1 \leqslant i \leqslant n+1, 1 \leqslant j \leqslant n). \quad (20)$$

The latter equation leads to

$$S_{i,j+1}^{(\ell)}(\lambda) = S_{i,1}^{(\ell-j)}(q^j\lambda) \qquad \ell \geqslant j. \quad (21)$$

For $\ell < j$,

$$S_{i,j}^{(\ell+1)}(\lambda) = S_{i,j-\ell}^{(1)}(q^\ell\lambda) \quad (22)$$

and the right-hand side is given directly in terms of $\tau^{(a)}$. Thus we regard only $S_{k,1}^{(\ell)}$ as non-trivial elements.

Equation (19) can be rewritten such that it contains only $S_{k,1}^{(\ell)}$ in the left-hand side. Let us write explicitly for $k = 1$,

$$S_{1,1}^{(\ell+1)}(\lambda) = \tau^{(1)}(\lambda)S_{1,1}^{(\ell)}(\lambda q) + \tau^{(2)}(\lambda)S_{1,1}^{(\ell-1)}(\lambda q^2) + \cdots + \tau^{(n+1)}(\lambda)S_{1,1}^{(\ell-n)}(\lambda q^{n+1}). \quad (23)$$

We remark on the properties of $S_{1,1}^{(\ell)}(\lambda)$ from the linear independence of the FSS. The (1, 1) component of equation (14) for $\ell = m + j$, $1 \leqslant j \leqslant n$ reads,

$$y_k = S_{1,1}^{(m+j)} y_{k+m+j} + \cdots.$$

Φ_{k+m+j} consists of $(y_{k+m+j}, y_{k+m+j+1}, \ldots, y_{k+m+n+j})$. As $y_{k+m+n+j} = (-1)^n y_{k+j-1}$, it contains $(-1)^n y_k$. Thus the linear independence of solutions concludes at least

$$S_{1,1}^{(m+j)} = 0 \qquad 1 \leqslant j \leqslant n. \quad (24)$$

We also note a trivial relation, $S_{1,1}^{(m+n+1)} = (-1)^n$.

In the next section, we will show that a set of solutions to the above recursion relations can be neatly represented by the quantum Jacobi–Trudi formula.

4. Quantum Jacobi–Trudi formula in Stokes multipliers

The recursion relation (23) can be successively solved for given initial conditions $\tau^{(1)}(\lambda), \ldots, \tau^{(n)}(\lambda)$ (forward propagation). Conversely, $\tau^{(a)}(\lambda)$ is expressible by these solutions (back propagation).

To represent these, we employ a notation which will be useful in later discussions,

$$T_\ell^{(1)}(\lambda) = S_{1,1}^{(\ell)}(\lambda q^{-(\ell-1)/2}) \quad (25)$$

$$T_1^{(a)}(\lambda) = (-1)^{a+1}\tau^{(a)}(\lambda q^{-(a-1)/2}) \qquad \left(= (-1)^{a+1}S_{a,1}^{(1)}(\lambda q^{-(a-1)/2})\right). \quad (26)$$

We first state the result for the forward-propagation problem,

$$T_{\ell+1}^{(1)}(\lambda) = \det\left(T_1^{(1-i+j)}(\lambda q^{(\ell+2-i-j)/2})\right)_{1\leqslant,i,j\leqslant \ell+1}. \quad (27)$$

The right-hand side includes the formal extension of $T_1^{(a)}(\lambda)$, and we understand $T_1^{(a)}(\lambda) = 0$ for $a \geqslant n+2$ or $a < 0$ and $T_1^{(0)}(\lambda) = 1$.

The proof is easy by induction on ℓ. The case $\ell = 0$ is trivial by definition. Assume that equation (25) is valid up to ℓ. Expanding the determinant with respect to the last column, we have

$$T^{(1)}_{\ell+1}(\lambda) = T^{(1)}_1(\lambda q^{-\ell/2})T^{(1)}_\ell(\lambda q^{1/2}) - T^{(2)}_1(\lambda q^{-(\ell-1)/2})T^{(1)}_{\ell-1}(\lambda q) + \cdots \\ +(-1)^{a+1}T^{(a)}_1(\lambda q^{-(\ell+1-a)/2})T^{(1)}_{\ell+1-a}(\lambda q^{a/2}) + \cdots. \tag{28}$$

Here we set $T^{(1)}_\ell(\lambda) = 0$, $\ell < 0$ and $T^{(1)}_0(\lambda) = 1$.

By equations (25), (26) and from the assumption of induction, the left-hand side is equal to that in equation (23) after $\lambda \to \lambda q^{-\ell/2}$. Then the validity of equation (25) carries over into $\ell + 1$, which completes the proof.

Next consider the back-propagation problem. We regard the coupled equations (23) for $\ell = 0, 1, 2, \ldots, n$, as a set of linear equations for $\tau^{(a)}(\lambda)$ (or $T^{(a)}_1(\lambda)$). Cramer's theorem is again applicable. Taking account of the fact that the determinant of the coefficient matrix is unity, we have

$$T^{(a)}_1(\lambda) = \det\big(T^{(1)}_{1-i+j}(\lambda q^{(i+j-a-1)/2})\big)_{1\leqslant,i,j\leqslant a}. \tag{29}$$

It is natural to define $T^{(0)}_1(\lambda) = 1$ from this expression.

We note the similarity of equations (27) and (29) to the Jacobi–Trudi formula for Schur functions [35]. It states that any complex Schur function associated with a skew Young diagram can be represented in terms of a determinant of a matrix whose elements are given by elementary Schur functions.

We find a quite parallel result in the present problem. Thanks to the argument around equations (21) and (22), only $S^{(\ell)}_{k,1}(\lambda)$ is of interest. This turns out to have a compact expression in a similar manner to the Jacobi–Trudi formula, which we call the quantum Jacobi–Trudi formula.

To state this, we prepare some notation. By μ, we mean a Young diagram $(\mu_1, \mu_2, \ldots)$ and by μ', its transpose. Consider a skew Young table of the shape $\mu/\eta = (\mu_1 - \eta_1, \mu_2 - \eta_2, \ldots)$ such that $\mu_i \geqslant \eta_i$. We define a quantity associated with μ/η,

$$T_{\mu/\eta}(\lambda) := \det{}_{1\leqslant j,k\leqslant \mu'_1}\big(T^{(1)}_{\mu_j-\eta_k-j+k}(\lambda q^{-(\mu'_1-\mu_1+\mu_j+\eta_k-j-k+1)/2})\big). \tag{30}$$

It contains equation (29) as a special case $\mu = (1, 1, \ldots, 1)$ and $\eta = \phi$.

A generalization of equation (27) also exists. To show this, consider two matrices H and E with entries

$$H_{i,j} := (-1)^{i-j}T^{(1)}_{i-j}(\lambda q^{-(i+j)/2})$$
$$E_{i,j} := T^{(i-j)}_1(\lambda q^{-(i+j)/2}).$$

Clearly, $\det H = \det E = 1$ as they are lower triangular with all diagonal elements unity. The expansion of equation (27) with respect to the first column leads to

$$0 = -T^{(1)}_{\ell+1}(\lambda) + T^{(1)}_1(\lambda q^{\ell/2})T^{(1)}_\ell(\lambda q^{-1/2}) - T^{(2)}_1(\lambda q^{(\ell-1)/2})T^{(1)}_{\ell-1}(\lambda q^{-1}) \\ + \cdots + (-1)^{a+1}T^{(a)}_1(\lambda q^{(\ell+1-a)/2})T^{(1)}_{\ell+1-a}(\lambda q^{-a/2}) + \cdots$$

which is equivalent to

$$\delta_{i,j} = \sum_k (-1)^{i-k}T^{(1)}_{i-k}(\lambda q^{-(i+k)/2})T^{(k-j)}_1(\lambda q^{-(k+j)/2}).$$

Hence E and H are inverse to each other.

These two facts are sufficient for the second representation of $T_{\mu/\eta}$ [28, 35, 39],

$$T_{\mu/\eta}(\lambda) = \det_{1\leqslant j,k\leqslant \mu_1}\left(T_1^{(\mu'_j-\eta'_k-j+k)}(\lambda q^{-(\mu'_1-\mu_1-\mu'_j-\eta'_k+j+k-1)/2})\right). \tag{31}$$

Equations (30) and (31) coincide with the Jacobi–Trudi formula by dropping the λ dependences.

We have the following statement on non-trivial Stokes multipliers.

Theorem 3. *The solution to the recursion relation (18) for $j = 1$ is given by*

$$S_{k,1}^{(\ell)}(\lambda) = (-1)^{k+1}T_\mu(q^{(\ell+k-2)/2}\lambda) \tag{32}$$

where μ is a Young diagram of hook shape $\mu = (\ell, 1, \dots, 1)$ with height k.

The proof is as follows. In terms of T, we need to show

$$T_{\mu'}(\lambda) + T_{\mu''}(\lambda) = T_1^{(k)}(\lambda q^{\ell/2})T_\ell^{(1)}(\lambda q^{-k/2})$$

where $\mu'(\mu'')$ is also the hook Young diagram $\mu' = (\ell + 1, 1, \dots, 1)$, $(\mu'' = (\ell, 1, \dots, 1))$ with height k $(k + 1)$. Then the equality is verified by expanding the determinant associated with $T_{\mu''}(\lambda)$ with respect to the first row.

It remains to show the consistency of equation (32) with equations (18) ($k = n + 1$ and $j = 1$) and (15).

First, consider $k = n + 1$ and $j = 1$ of equation (18). By the recursion argument in the above, the formal extension $S_{n+2,1}^{(\ell)}(\lambda)$, defined by the right-hand side of equation (32), must be zero. This is directly seen from the second expression (31), as $T_1^{(a)}(\lambda) = 0$ $(a \geqslant n + 2)$.

Next we deal with equation (15). To be precise we will show

$$S_{i,j}^{(m+n+1)}(\lambda) = T_{m+n+1}^{(1)}(\lambda q^{(m+n)/2})\delta_{i,j} \quad (= (-1)^n\delta_{i,j}). \tag{33}$$

With a slight redefinition of index j and the application of the relation (21), this is converted into an equivalent form,

$$S_{i,1}^{(m+j+1)}(\lambda) = T_{m+n+1}^{(1)}(\lambda q^{(m+n)/2})\delta_{i+j,n+1} \qquad (0 \leqslant j \leqslant n). \tag{34}$$

We shall divide the argument into two cases, $j \neq 0$ and $j = 0$.

For $j = 0$, we apply equation (32) to $S_{i,1}^{(m+1)}(\lambda)$. Then we find from formula (30) that all the elements of the first row are zero unless $i = n + 1$. This comes from equation (24), $T_{m+j'}(\lambda) = 0$ for $1 \leqslant j' \leqslant n$. For $i = n + 1$, we expand the determinant with respect to the last column. From the normalization in equation (32) and $T_0^{(1)}(\lambda) = 1$, it follows that $S_{n+1,1}^{(m+1)}(\lambda) = T_{m+n+1}^{(1)}(\lambda q^{(m+n)/2})$, which is $(-1)^n$ by definition.

A remark is in order. The second form (31) leads to $S_{n+1,1}^{(m+1)}(\lambda) = (-1)^n T_m^{(1)}(\lambda q^{(m-1)/2})$ $T_1^{(n+1)}(\lambda q^{3m+n/2})$. Then

$$T_m^{(1)}(\lambda q^{-(n+1)/2})T_1^{(n+1)}(\lambda q^{m/2}) = (-1)^n T_{m+n+1}^{(1)}(\lambda) = 1 \tag{35}$$

must hold. Neither $T_m^{(1)}(\lambda)$ or $T_1^{(n+1)}(\lambda)$ has poles. Thus equation (35) asserts that they are also non-zero everywhere. Consequently they are constant. We drop their λ dependences for the time being. By taking the determinants of both sides of (the matrix form of) equation (33), one obtains $(T_1^{(n+1)})^{m+n+1} = (T_{m+n+1}^{(1)})^{n+1}$. Combining with equation (35), this yields, $(T_1^{(n+1)})^m = (T_m^{(1)})^{n+1}$. We conclude that both $T_1^{(n+1)}$ and $T_m^{(1)}$ are roots of unity. The former can be derived directly once if we assume (5) and use (11). One only has to evaluate W_k using asymptotic forms (4) and (6) in $\mathcal{S}_{k+(n+1)/2} \cup \mathcal{S}_{k+(n-1)/2}$. The simple manipulation yields $\tau_k^{(n+1)} = (-1)^n$, thus $T_1^{(n+1)} = 1$.

For $j \neq 0$, the recursion (18) yields

$$S_{i,1}^{(m+j+1)}(\lambda) = \tau^{(i)}(\lambda q^{m+j}) S_{1,1}^{(m+j)}(\lambda) + S_{i+1,1}^{(m+j)}(\lambda)$$
$$= (-1)^i T^{(i)}(\lambda q^{m+j+(i-1)/2}) T_{m+j}^{(1)}(\lambda q^{(m+j-1)/2}) + S_{i+1,1}^{(m+j)}(\lambda). \qquad (36)$$

In the present case, the first term in equation (36) is vanishing due to equation (24). Thus

$$S_{i,1}^{(m+j+1)}(\lambda) = S_{i+1,1}^{(m+j)}(\lambda). \qquad (37)$$

We then try to prove equation (34) by induction on the upper index, starting from the result for $S_{i,1}^{(m+1)}(\lambda)$ as the 'initial condition'. In each induction step, however, the $i = n+1$ component is indeterminate from equation (37). We then use equation (31) and find $S_{n+1,1}^{(\ell)}(\lambda) = (-1)^{\ell+1} T_{\ell-1}^{(1)}(\lambda q^{\omega(\ell)}) T_1^{(n+1)}(\lambda q^{\omega'(\ell)})$ where $\omega(\ell)$ and $\omega'(\ell)$ are some shifts which are irrelevant in the present argument. This supplies the missing pieces. As $\ell \geqslant m+2$ for the case under consideration, these are null. We then immediately check that equation (34) holds.

Thereby, we prove that non-trivial Stokes multipliers are given explicitly via formula (32). The fundamental quantities $\tau^{(a)}$ are not all independent. The constraints (24) impose complex algebraic equations among them.

We note useful relations among Ts, the T-system. Let $T_\ell^{(a)}(\lambda) = T_\mu(\lambda)$, where μ is a rectangle of height a and width ℓ. By applying the Plücker relation to equation (30) or (31), one finds,

$$T_\ell^{(a)}(\lambda q^{1/2}) T_\ell^{(a)}(\lambda q^{-1/2}) = T_{\ell+1}^{(a)}(\lambda) T_{\ell-1}^{(a)}(\lambda) + T_\ell^{(a+1)}(\lambda) T_\ell^{(a-1)}(\lambda) \qquad a = 1, \ldots, n. \qquad (38)$$

The boundary conditions are $T_{\ell<0}^{(1)}(\lambda) = T_{m+1\leqslant\ell\leqslant m+n}^{(1)}(\lambda) = T_1^{(a<0)}(\lambda) = T_1^{(a>n+1)}(\lambda) = 0$ and $T_\ell^{(a)}(\lambda) = 1$ if $a = 0$ or $\ell = 0$. The last relation comes from the null dimensionality of the matrix in equations (30) and (31). The initial condition $T_{\ell<0}^{(1)}(\lambda) = 0$ leads to $T_{\ell<0}^{(a)}(\lambda) = 0$ $(1 \leqslant a \leqslant n)$. Similarly, we have a $T_{m+j}^{(a)}(\lambda) = 0$ $(1 \leqslant a, j \leqslant n)$. Thus the T-system constitutes a *finite* number of relations with a *finite* number of $T_\ell^{(a)}(\lambda)$.

Note that the case with $n = 1$, $m = 3$ and with multi-parameters has been found in [34] by a direct calculation of recursion relations. The one-parameter result was later extended to arbitrary m by a similar argument to that employed here [2].

Summarizing this section, we have found determinant representations of Stokes multipliers. We conveniently introduce a subset of huge hierarchy, $T_\ell^{(a)}(\lambda)$ including (the most fundamental) Stokes multipliers $\tau^{(a)}(\lambda)$. Then functional relations exist among them which, in turns, impose some restrictions on $\tau^{(a)}(\lambda)$.

We will discuss the above results in view of solvable models in the next section. In particular, we will give some follow-up to the last sentence in the previous paragraph.

5. Functional relations in solvable models and the thermodynamic Bethe ansatz

Hereafter we assume n odd and $m = M(n+1)$. It is then convenient to rotate the λ for the differential equation (1) by $\frac{1}{2}\pi$. We will use the same notation for the resultant λ.

In the first part of this section, we remind the reader of some concrete results in solvable models. The second part is devoted to a rather speculative discussion of the possibility of the application of the thermodynamic Bethe ansatz to the evaluation of Stokes multipliers.

The commuting transfer matrices play a fundamental role in studies of solvable lattice models and field theories [11–13]. The members in a commuting family share the same physical space (quantum space) and are parametrized by the (multiplicative) spectral parameter λ. They are labelled by the auxiliary space of which the trace must be taken. These auxiliary spaces are identified with irreducible modules of Yangian or quantum affine Lie algebra [37, 38]. For $U_q(A_n^{(1)})$, there exists a irreducible module $W_m^{(a)}(\lambda)$ $(a \leqslant n)$ which is isomorphic to mV_{Λ_a} as a classical module. Naturally, we associate a Young diagram of $a \times m$ to this module, and write the corresponding transfer matrix $\mathcal{T}_m^{(a)}(\lambda)$. Since they are commutative, we consider transfer matrices on common eigenstates. Thus we sometimes do not distinguish operators from their eigenvalues.

In the language of solvable lattice models, $\mathcal{T}_m^{(a)}(\lambda)$ is a transfer matrix of a model obtained by the fusion procedure. Starting from a 'fundamental' model acting on $V_{\Lambda_1} \times V_{\Lambda_1}$, we can derive fusion models recursively. Their auxiliary spaces are constructed from $V_{\Lambda_1} \times \cdots \times V_{\Lambda_1}$ by applying appropriate projectors. We utilize singular points of the R-matrix to construct these projectors.

The fusion procedure is not restricted to the rectangle type. Generally, there exists a module $W_{\mu/\eta}(\lambda)$, or a fusion model, parametrized by a skew Young diagram μ/η. We write the corresponding transfer matrix as $\mathcal{T}_{\mu/\eta}(\lambda)$.

The short exact sequence of irreducible modules examined in [28, 29, 36] leads to the relation which we have seen in equation (30),

$$\mathcal{T}_{\mu/\eta}(\lambda) = \det_{1\leqslant j,k\leqslant \mu_1}\big(\mathcal{T}_1^{(\mu'_j-\eta'_k-j+k)}(\lambda q^{-(\mu'_1-\mu_1-\mu'_j-\eta'_k+j+k-1)/2})\big) \tag{39}$$

$$= \det_{1\leqslant j,k\leqslant \mu'_1}\big(\mathcal{T}^{(1)}_{\mu_j-\eta_k-j+k}(\lambda q^{-(\mu'_1-\mu_1+\mu_j+\eta_k-j-k+1)/2})\big) \tag{40}$$

where $\mathcal{T}^{(1)}_{m<0}(\lambda) = \mathcal{T}_1^{(a<0)}(\lambda) = 0$.

This ensures the T-system (38) among $\mathcal{T}$ [29]. The quantum Jacobi–Trudi formula plays a role in several problems in solvable models [39–41]

The conditions (24) also hold. When $q = \exp\big(\mathrm{i}\frac{2\pi}{m+n+1}\big)$, the truncation of the space happens due to quantum group symmetry. In view of solvable lattice models, it corresponds to the situation that some projectors are vanishing and fusion paths in local variables are lost. Consequently, no local variables can be adjacent and transfer matrices vanish [28],

$$\mathcal{T}^{(a)}_{m+1}(\lambda) = \mathcal{T}^{(a)}_{m+2}(\lambda) = \cdots = \mathcal{T}^{(a)}_{m+n}(\lambda) = 0. \tag{41}$$

The normalizations of $\mathcal{T}_0^{(a)}(\lambda)$, $\mathcal{T}_\ell^{(0)}(\lambda)$ and $\mathcal{T}_\ell^{(n+1)}(\lambda)$ depend on the choice of quantum space. We have not yet found the description of the quantum space yielding $\mathcal{T}_0^{(a)}(\lambda) = \mathcal{T}_\ell^{(0)} = \mathcal{T}_\ell^{(n+1)} = 1$, except for $n = 1$. We assume its existence for general n.

Then, on the corresponding space, the fusion transfer matrices share same functional relations with Stokes multipliers.

Below we use the same symbol $T_\ell^{(a)}(\lambda)$ for these two cases.

Thanks to the (quantum group) reduction, the T-system closes within a finite set of unknowns.

The solution to functional relations, however, is not unique. Additional information on the analyticity of $T_\ell^{(a)}(\lambda)$ in complex λ is required for the uniqueness.

We start from the lattice model. Then we take the scaling limit (or the field-theoretical limit) which has been discussed in several papers. In the present context, we refer to [42]. Note that our transfer matrices are not mere sums of products of Boltzmann weights but are imposed overall renormalizations due to $\mathcal{T}_0^{(a)}(\lambda) = \mathcal{T}_\ell^{(0)} = \mathcal{T}_\ell^{(n+1)} = 1$.

Let us define an 'additive' spectral parameter u by $\lambda = \exp\left(\frac{\pi u}{(m+n+1)}\right)$. We also introduce

$$Y_\ell^{(a)}(u) := \frac{T_{\ell+1}^{(a)}(\lambda(u))T_{\ell-1}^{(a)}(\lambda(u))}{T_\ell^{(a+1)}(\lambda(u))T_\ell^{(a-1)}(\lambda(u))} \qquad (1 \leqslant a \leqslant n, 1 \leqslant \ell \leqslant m).$$

Then equation (38) reads

$$Y_\ell^{(a)}(u+i)Y_\ell^{(a)}(u-i) = \frac{(1+Y_{\ell+1}^{(a)}(u))(1+Y_{\ell-1}^{(a)}(u))}{(1+(Y_\ell^{(a+1)}(u))^{-1})(1+(Y_\ell^{(a-1)}(u))^{-1})} \tag{42}$$

where $(Y_\ell^{(0)}(u))^{-1} = Y_m^{(a)}(u) = 0$ by definition.

Remember that Boltzmann weights are regular functions of u. Then, apart from some renormalization factors mentioned above, $T_\ell^{(a)}(\lambda(u))$ must not be singular in the strip $\mathrm{Im}\, u \in [-1, 1]$ but may possess zeros in general.

Suppose that $T_\ell^{(a)}(\lambda(u))$ has finitely many distinct zeros in the strip $\{u_{\ell,k}^{(a)}\}$ which depend on eigenstates. They are assumed to be off the lines $\mathrm{Im}\, u = \pm 1$.

Then equation (42) can be transformed into coupled nonlinear integral equations by the standard trick [12, 13, 17, 43–45].

Both sides of equation (42) are analytic with finitely many distinct zeros and have constant asymptotics. Then, applying Cauchy's theorem, we have

$$\log Y_\ell^{(a)}(u) = \log \mathcal{Z}_\ell^{(a)}(u) + \sum_{b=1}^{a}\sum_{r=1}^{m}\int_{-\infty}^{\infty} K(u-u') \times \log \frac{(1+Y_{\ell+1}^{(a)}(u'))(1+Y_{\ell-1}^{(a)}(u'))}{(1+(Y_\ell^{(a+1)}(u'))^{-1})(1+(Y_\ell^{(a-1)}(u'))^{-1})} \mathrm{d}u' + C_\ell^{(a)} \tag{43}$$

where $C_\ell^{(a)}$ is a 'integral' constant fixed by comparing asymptotic values of both sides.

$\mathcal{Z}_\ell^{(a)}(u)$ signifies,

$$\begin{aligned} \mathcal{Z}_\ell^{(a)}(u) &= \frac{Z_{\ell+1}^{(a)}(u)Z_{\ell-1}^{(a)}(u)}{Z_\ell^{(a+1)}(u)Z_\ell^{(a-1)}(u)} \\ Z_\ell^{(a)}(u) &= \prod_j (\tanh \tfrac{1}{4}\pi(u-w_{\ell,j}^{(a)}))^{\epsilon_{\ell,j}} \prod_k \tanh \tfrac{1}{4}\pi(u-u_{\ell,k}^{(a)}) \end{aligned} \tag{44}$$

and $Z_\ell^{(0)}(u) = Z_\ell^{(n+1)}(u) = Z_0^{(a)}(u) = 1$.

Here $\{w_{\ell,j}^{(a)})\}$ is the joint set of zeros ($\epsilon_{\ell,j} = 1$) and singularities ($\epsilon_{\ell,j} = -1$) of $T_\ell^{(a)}(u)$ in $\mathrm{Im}\, u \in [-1, 1]$ due to the above normalization. They stem from common factors in Boltzmann weights, thus they are zeros or singularities of order N (where N is the system size). Here we label them disregarding their multiplicities.

The kernel function is easily written in Fourier-transformed form. We define the Fourier transformation $\widehat{f}[k]$ of a function $f(u)$ by

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}[k]\,\mathrm{e}^{\mathrm{i}ku}\,\mathrm{d}u \qquad \widehat{f}[k] = \int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{\mathrm{i}ku}\,\mathrm{d}u$$

then

$$\widehat{K}[k] = \frac{1}{2\cosh k}.$$

Equation (43) might be represented by the following form, in analogy with the thermodynamic Bethe ansatz equation in CFT,

$$\log \tilde{Y}_\ell^{(a)}(u) = \sum_{r=1}^{m} \int_{-\infty}^{\infty} A^{\ell,r}(u-u') \log \mathcal{Z}_r^{(a)}(u')\, \mathrm{d}u' + \sum_{b=1}^{a} \sum_{r=1}^{m} \int_{-\infty}^{\infty} K_{a,b}^{\ell,r}(u-u') \log(1+(Y_r^{(b)}(u'))^{-1})\, \mathrm{d}u' + D_\ell^{(a)} \qquad (45)$$

where $D_\ell^{(a)}$ is also some 'integral' constant and $\tilde{Y}_\ell^{(a)}(u) = Y_\ell^{(a)}(u)/Z_\ell^{(a)}(u)$.

The Fourier transformations of the kernel functions read,

$$\widehat{K}_{a,b}^{\ell,r}[k] = \widehat{A}^{\ell,r}[k]\widehat{M}_{a,b}[k]$$

$$\widehat{A}^{\ell,r}[k] = \frac{\sinh(\min(\ell,r)k)\sinh((m-\max(\ell,r))k)}{\sinh(mk)\sinh(k)}$$

$$\widehat{M}_{a,b}[k] = 2\cosh k\left(\delta_{a,b} - \frac{I_{a,b}}{2\cosh k}\right)$$

and $I_{a,b} = 1$ if they are on adjacent nodes of the Dynkin diagram for A_n and $I_{a,b} = 0$ otherwise.

At this stage, one performs the field-theoretical limit, system size $\to \infty$, lattice spacing $\to 0$, and sending elliptic nome $\to 0$ under some fine-tuning condition. Precisely speaking, this condition depends on the regime of the lattice model. We shall skip that detail here. We refer to [42] for details in the case of $A_1^{(1)}$.

Then it can be shown that $\{w_{\ell,j}^{(a)}\}$ accumulate at ∞ and reproduce 'momentum' term in equation (45) from the first term [1, 12, 13, 15, 42],

$$\log \tilde{Y}_\ell^{(a)}(u) = M_\ell^{(a)} \exp\left(\frac{\pi}{m}u\right) + \sum_{r=1}^{m} \int_{-\infty}^{\infty} A^{\ell,r}(u-u') \log \mathcal{Z}'^{(a)}_r(u')\, \mathrm{d}u' + \sum_{b=1}^{a} \sum_{r=1}^{m} \int_{-\infty}^{\infty} K_{a,b}^{\ell,r}(u-u') \log(1+(Y_r^{(b)}(u'))^{-1})\, \mathrm{d}u' + D_\ell^{(a)}. \qquad (46)$$

$\mathcal{Z}'^{(a)}_r(u')$ consists only of $\{u_{\ell,k}^{(a)}\}$ in (44).

The rapidity θ in [1] is denoted by $\frac{\pi}{m}u$ here.

The resultant coupled integral equations can be solved recursively and yield a unique set of solutions to $Y_\ell^{(a)}(u)$, and then to $T_\ell^{(a)}(\lambda(u))$ for given $\{u_{\ell,k}^{(a)}\}$ and $M_\ell^{(a)}$. Thus one must adopt an appropriate choice for these parameters to reproduce proper $T_\ell^{(a)}(\lambda(u))$ including $\tau^{(a)}(\lambda)$. Conversely, if two sets of functions satisfy the same relation (38) and share the same $\{u_{\ell,k}^{(a)}\}$ and $M_\ell^{(a)}$, then they are identical in the strip.

For solvable lattice models, the case studies on $U_q(A_{1,2,3}^{(1)})$ indicate that $T_\ell^{(a)}(\lambda(u))$ does not possess zeros in the strip $\mathrm{Im}\, u \in [-1, 1]$ when acting on the largest eigenvalue sector. That is, $\{u_{\ell,k}^{(a)}\} = \phi, a = 1, \ldots, n$ for all $p \leqslant m$.

This seems to also be the case with Stokes multipliers. Actually this is the case for $n = 1$. That is, the proper Stokes multipliers are reproduced by $\{u_{\ell,k}^{(a)}\} = \phi, a = 1, \ldots, n$ for all $p \leqslant m$ but a different choice of $M_\ell^{(a)}$ from lattice models or the field-theoretic model. Let me just present an argument for the $n \geqslant 2$ case. Assume that the solution to equation (35) is simple, $T_1^{(n+1)} = T_m^{(1)} = 1$. As remarked above, the former equality is a direct consequence of (5) and (11). From our 'boundary conditions', the T-system is then invariant under the simultaneous transformations $(a, \ell) \to (n+1-a, m-\ell)$. This implies $T_\ell^{(a)}(\lambda) = T_{m-\ell}^{(n+1-a)}(\lambda)$. By $\pm\theta_\ell^{(a)}$, we mean the imaginary part of the zero of $T_\ell^{(a)}(\lambda)$. Then a sum rule holds,

$$\theta_\ell^{(a)} + \theta_{m-\ell}^{(n+1-a)} = m + n + 1.$$

The simplest solution, $\theta_\ell^{(a)} = a + \ell$, is actually correct for $n = a = 1$. Postulating this solution, we also conclude $\{u_{\ell,k}^{(a)}\} = \phi, a = 1, \ldots, n$ for all $p \leqslant m$ for Stokes multipliers.

We do not expect ill-behaviour for a small-angle sector of λ; $\tau^{(a)}(\lambda)$ cannot be vanishing in the sector otherwise it harms the linear dependency of FSS. The above choice of $\theta_\ell^{(a)}$ is consistent with this expectation.

Admitting these assumptions, we have a conjecture

Conjecture 1. *The Stokes multipliers of equation (1) are given by 'vacuum expectation values' of fusion transfer matrices related to* $U_q(A_n^{(1)})$. *Thus they are evaluated from equation (46) by forgetting the second term in the right-hand side.*

The parameters $M_\ell^{(a)}$ and $D_\ell^{(a)}$ must be tuned correctly as in [1] for $n = 1$.

To be precise, overall normalization of λ for the identification is not fixed by functional relations only. For $n = 1$ the determination of this factor is quite straightforward [1, 2], due to the fact that only one subdominant solution exists in each $\mathcal{S}_j$. The spectral determinant is then identified with the fusion Stokes multiplier which connects the subdominant solution on the negative real axis to the dominant solution on the positive real axis. Then standard WKB arguments yield the identification of parameters, in particular, $M_\ell^{(1)}$ and the normalization of λ.

This cannot be generalized straightforwardly for $n \geqslant 2$. Let me just comment on two fundamental problems. First, the meaning of the eigenvalue problem is not necessary clear for higher-order differential equations. The characterization of the eigenspace (it is the Hilbert space for $n = 1$) is not obvious. Second, more technically, there are several subdominant solutions in each sector. We hope to clarify these issues in future publications.

Finally, we comment on the (1, 1) component of equation (8),

$$y_k = \tau_k^{(1)} y_{k+1} + \tau_k^{(2)} y_{k+2} + \cdots + \tau_k^{(n+1)} y_{k+n+1}. \tag{47}$$

The equation coincides with the Baxter's $T - Q$ relation for $n = 1$ by setting $x = 0$ and writing $y_k(x = 0) = Q(\lambda q^{k+\text{constant}})$. For general n, this coincides with the 'spectral curve' equation argued in [46] or the characteristic equation in the quantum separation of variables [47].

6. Discussion

We have seen that Stokes multipliers associated with an $(n+1)$th differential equation satisfy the same recursion relations with fusion transfer matrices related to $U_q(A_n^{(1)})$. Under the assumption on analyticity of some functions in the strip, we conjecture thermodynamic Bethe ansatz-type equations (46) that determine Stokes multipliers. The assumption, however, needs an extensive numerical check, which we hope to report on in the near future.

The deformation parameter $q = \exp\left(\mathrm{i}\frac{2\pi}{m+n+1}\right)$, which arises naturally in the present context, has a concrete meaning in the solvable models. The parameter m specifies the level of dominant integral weight of the local variables in the lattice model [48]. Then the denominator of the exponent of q is of the form, level + dual Coxeter number. This combination also appears as the deformation parameter of a solvable model based on other affine Lie algebras [49]. We thus expect a possible extension of the present study to other types of potential terms related to other affine Lie algebras.

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Appendix

We present the argument leading to equation (5).

In terms of the vector $u := (y, \partial y, \ldots, \partial^n y)$, equation (2) reads,

$$\frac{\mathrm{d}}{\mathrm{d}x}u = A(x)u = \begin{pmatrix} 0, & 1, & 0, & \ldots, & 0 \\ 0, & 0, & 1, & \ldots, & 0 \\ \vdots & & & & \vdots \\ 0, & 0, & 0, & \ldots, & 1 \\ P(x), & 0, & 0, & \ldots, & 0 \end{pmatrix}.$$

We define a new variable ξ by $\xi^{n+1} = x$. Put $u = MTw$ such that $M = \mathrm{diag}(1, \xi^m, \xi^{2m}, \ldots)$ and that T diagonalizes a matrix of the form $A(x)$ in the above with $P(x) = 1$. Then the equivalent matrix equation is written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}w = \xi^{m+n}B(\xi)w = \xi^{m+n}\sum_{j=0} B_j\xi^{-j}w \tag{A1}$$

and $B_0 = \mathrm{diag}(\exp(\mathrm{i}\frac{2\pi n}{n+1}), \exp(\mathrm{i}\frac{2\pi(n-1)}{n+1}), \ldots, 1)$.

Following [33, 34], we assume w in the form,

$$w = \begin{pmatrix} p_1(\xi) \\ p_2(\xi) \\ \vdots \\ p_{n+1}(\xi) \end{pmatrix} \exp\left(\int^{\xi} \eta^{m+n}\gamma(\eta)\,\mathrm{d}\eta\right)$$

and $p_1(\xi) = 1$.

Then the substitution of the above form into equation (A1) yields,

$$\xi^{-m-n}\frac{\mathrm{d}}{\mathrm{d}\xi}p_j = \beta_{j,1} + (\beta_{j,j} - \beta_{1,1})p_j - \sum_{i\neq 1}\beta_{1,i}p_ip_j + \sum_{k\neq 1,j}\beta_{j,k}p_k \tag{A2}$$

where $\beta_{j,k}$ denotes the (i, j)th component of $B(\xi)$ and $j \geqslant 2$. Note that $\beta_{j,k} = \mathcal{O}(1/\xi)$ if $j \neq k$.

We can immediately see that $\widehat{p}_j(\xi) = \sum_{N=1} p_{j,N}\xi^{-N}$ is a formal solution; one can recursively determine $p_{j,N}$ by (A2).

We define $h_0 = \beta_{n+1,n+1}(\infty) - \beta_{1,1}(\infty)$, and the sector $\mathcal{S}$ in the ξ-plane by

$$|\arg h_0 + (m+n+1)\arg\xi| \leqslant \tfrac{3}{2}\pi \qquad |\xi| \geqslant \Omega$$

for a fixed positive Ω. By $\mathcal{D}_r$ we mean the domain in $(a_1, \ldots, a_m)$ space such that

$$|a_1|^2 + \cdots + |a_m|^2 < r$$

for some positive r. Then we naturally generalize the result in [33, 34].

Conjecture 2. *One can construct* $\widehat{p_{j,r}}(\xi)$ *so that* $\widehat{p_{j,r}}(\xi)$ *is holomorphic with respect to* $(\xi, a_1, \ldots, a_m)$ *in* $\mathcal{S} \times \mathcal{D}_r$ *and* $\partial^k \widehat{p_{j,r}}(\xi)$ $(k = 0, 1, \ldots)$ *admits the uniformly asymptotic expansions,*

$$\partial^k \widehat{p_{j,r}}(\xi) \sim \partial^k \widehat{p}_j(\xi)$$

for $(a_1, \ldots, a_m) \in \mathcal{D}_r$ *as* ξ *tends to infinity in* $\mathcal{S}$.

Let us present some argument supporting this (by no means a proof). We put $p_j = q_j + \widehat{p_{j,r}}(\xi)$ and inserting this into (A2),

$$\xi^{-m-n}\frac{\mathrm{d}}{\mathrm{d}\xi}q_j = C_{j,1} + C_{j,k}q_j - \sum_{i\neq 1}\beta_{1,i}q_iq_j \qquad (j \geqslant 2)$$

$$C_{j,1} = \beta_{j,1} + (\beta_{j,j} - \beta_{1,1})\widehat{p_{j,r}}(\xi) + \sum_{k\neq 1,j}\beta_{j,k}\widehat{p_{k,r}}(\xi)$$
$$\qquad - \sum_{i\geqslant 2}\beta_{1,i}\widehat{p_{i,r}}(\xi)\widehat{p_{j,r}}(\xi) - \xi^{-m-n}\frac{\mathrm{d}}{\mathrm{d}\xi}\widehat{p_{j,r}}(\xi) \qquad (j \geqslant 2)$$

$$C_{j,k} = \beta_{j,k}(1-\delta_{j,k}) + \delta_{j,k}\{(\beta_{j,j} - \beta_{1,1}) - \sum_{i\geqslant 2}\beta_{1,i}\widehat{p_{i,r}}(\xi)\} - \beta_{1,k}\widehat{p_{j,r}}(\xi) \qquad (j, k \geqslant 2).$$

The formal solution is

$$q_j(\xi) = \int^{\xi}\eta^{m+n}\left(C_{j,1} + \phi_{j,k}q_k + \sum_{i\neq 1}\beta_{1,i}q_iq_j\right)\exp\left(\frac{h_0^{(j)}}{m+n+1}(\xi^{m+n+1} - \eta^{m+n+1})\right)\mathrm{d}\eta \tag{A3}$$

where we set $C_{j,k} = h_0^{(j)}\delta_{j,k} + \phi_{j,k}$ and $h_0^{(j)} = (n+1)\big(\exp\big(2\pi\mathrm{i}\frac{n+1-j}{n+1}\big) - \exp\big(2\pi\mathrm{i}\frac{n}{n+1}\big)\big)$. Now the coefficients in the brackets of the integrand are less than order $1/\xi$.

We are interested in equation (4). For this, it may be enough to treat $j = n+1$. Assume that there exists an appropriate choice of M, then lemma 3 of [33] also applies in this case. The integral in the right-hand side of (A3) can be thus bounded from above, $|q_{n+1}| \leqslant \mathcal{O}(|\xi|^{-1})$ in a certain domain provided that

$$|\arg h_0^{(n+1)} + (m+n+1)\arg\xi| < \tfrac{3}{2}\pi - \epsilon \tag{A4}$$

where ϵ is positive and small. By definition, $h_0^{(n+1)} = h_0$, thus in $\mathcal{S}$, q_{n+1} tends to zero as $|\xi| \to \infty$. Hence the solution converges to the asymptotic expansion form. It may be shown that the limit does not depend on r as in [33, 34]. The condition (A4) coincides, when written in terms of x, with (5).

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